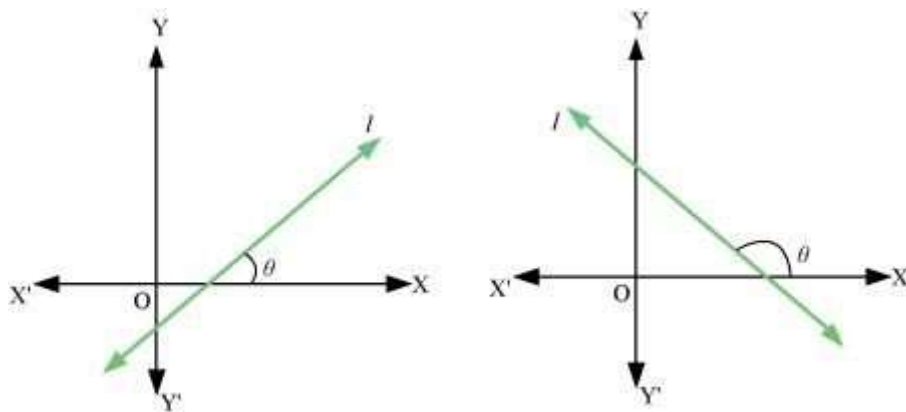


5. Straight Line

- **Slope of a line:** If θ is the inclination of a line l (the angle between positive x -axis and line l), then $m = \tan \theta$ is called the slope or gradient of line l .



- The slope of a line whose inclination is 90° is not defined. Hence, the slope of the vertical line, y -axis is undefined.
- The slope of the horizontal line, x -axis is zero.

For example, the slope of a line making an angle of 135° with the positive direction of x -axis is $m = \tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1$

- **Slope of line passing through two given points:**

The slope (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2.$$

For example, the slope of the line joining the points $(-1, 3)$ and $(4, -2)$ is given by,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 3}{4 - (-1)} = -\frac{5}{5} = -1$$

- **Conditions for parallelism and perpendicularity of lines:**

Suppose l_1 and l_2 are non-vertical lines having slopes m_1 and m_2 respectively.

- l_1 is parallel to l_2 if and only if $m_1 = m_2$ i.e., their slopes are equal.
- l_1 is perpendicular to l_2 if and only if $m_1 m_2 = -1$ i.e., the product of their slopes is -1 .

Example:

Find the slope of the line which makes an angle of 45° with a line of slope 3.

Solution:

Let m be the slope of the required line.

$$\therefore \tan 45^\circ = \left| \frac{m-3}{1+3m} \right|$$

$$\Rightarrow \left| \frac{m-3}{1+3m} \right| = 1$$

$$\Rightarrow \left| \frac{m-3}{1+3m} \right| = \pm 1$$

$$\Rightarrow \frac{m-3}{1+3m} = 1 \quad \text{or} \quad \frac{m-3}{1+3m} = -1$$

$$\Rightarrow m-3 = 1+3m \quad \text{or} \quad m-3 = -1-3m$$

$$\Rightarrow -2m = 4 \quad \text{or} \quad 4m = 2$$

$$\Rightarrow m = -2 \quad \text{or} \quad m = \frac{1}{2}$$

- **Collinearity of three points:** Three points A, B and C are collinear if and only if slope of AB = slope of BC

- The equation of a horizontal line at distance a from the x -axis is either $y = a$ (above x -axis) or $y = -a$ (below x -axis).
- The equation of a vertical line at distance b from the y -axis is either $x = b$ (right of y -axis) or $x = -b$ (left of y -axis).

• Point-slope form of the equation of a line

The point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) if and only if its coordinates satisfy the equation. This means $y - y_0 = m(x - x_0)$.

Example: Find the equation of the line passing through $(4, 5)$ and making an angle of 120° with the positive direction of x -axis?

Solution: Slope of the line, $m = \tan 120^\circ = \tan (180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}$

Equation of the required line is,

$$y - 5 = -\sqrt{3}(x - 4)$$

$$\Rightarrow y - 5 = -\sqrt{3}x + 4\sqrt{3}$$

$$\Rightarrow \sqrt{3}x + y - (5 + 4\sqrt{3}) = 0$$

• Two-point form of the equation of a line

The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

Example: Find the equation of the line passing through the points $(-5, 2)$ and $(1, 6)$.

Solution: Equation of the line passing through points $(-5, 2)$ and $(1, 6)$ is

$$\begin{aligned}
 y - 2 &= \frac{6-2}{1-(-5)}(x - (-5)) \\
 \Rightarrow y - 2 &= \frac{4}{6}(x + 5) \\
 \Rightarrow y - 2 &= \frac{2}{3}(x + 5) \\
 \Rightarrow 3y - 6 &= 2x + 10 \\
 \Rightarrow 2x - 3y + 16 &= 0
 \end{aligned}$$

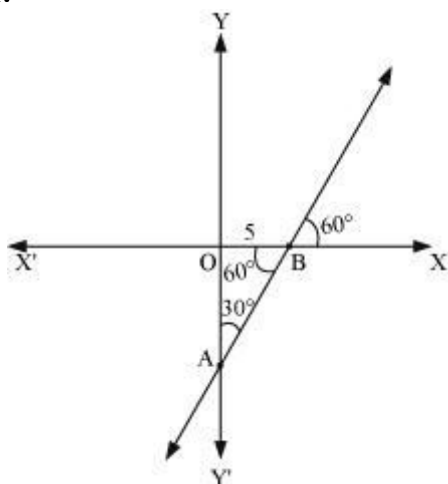
• **Slope-intercept form of a line**

- The equation of the line, with slope m , which makes y -intercept c is given by $y = mx + c$.
- The equation of the line, with slope m , which makes x -intercept d is given by $y = m(x - d)$.

Example:

Find the equation of the line which cuts off an intercept 5 on the x -axis and makes an angle of 30° with the y -axis.

Solution:



Slope of the line, $m = \tan 60^\circ = \sqrt{3}$

OB = 5

Intercept on the x -axis, $c = -OB = -5$ and $\tan 60^\circ = \frac{y}{x}$

Equation of the required line is $y = \sqrt{3}x + (-5\sqrt{3})$.

• **General equation of line**

Any equation of the form $Ax + By + C = 0$, where A and B are not zero simultaneously is called the general linear equation or general equation of line.

Slope of the line = $-\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{A}{B}$

y -intercept = $-\frac{C}{B}$

Example:

Find the slope and the y -intercept of the line $2x - 3y = -16$.

Solution:

The equation of the given line can be rewritten as $2x - 3y + 16 = 0$.

Here, $A = 2$, $B = -3$ and $C = 16$.

$$\text{Slope of the line} = -\frac{A}{B} = -\frac{2}{(-3)} = \frac{2}{3}$$

$$\text{Intercept on the } y\text{-axis} = -\frac{C}{B} = -\frac{16}{(-3)} = \frac{16}{3}$$

- **Intercept form**

The equation of the line making intercepts a and b on x -axis and y -axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$

Example:

If a line passes through $(3, 2)$ and cuts off intercepts on the axes in such a way that the product of the intercepts is 24, then find the equation of the line.

Solution:

The equation of a line in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots(1)$$

Where, a and b are the intercepts on x and y axes respectively.

Since the line passes through $(3, 2)$, we obtain

$$\frac{3}{a} + \frac{2}{b} = 1$$

$$\Rightarrow 3b + 2a = ab$$

$$\Rightarrow 2a + 3b = 24 \quad \dots(2) \quad (\text{Since product of intercepts is given as 24})$$

Now,

$$(2a - 3b)^2 = 24ab$$

$$= (24)^2 - 24(24) \quad [\text{From equation (2)}]$$

$$= 0$$

$$\therefore 2a - 3b = 0 \quad \dots(3)$$

On adding equations (2) and (3), we obtain

$$4a = 24 \Rightarrow a = 6$$

$$\therefore 3b = 2a = 2 \times 6 = 12$$

$$\Rightarrow b = 4$$

Hence, from (1), the required equation of line is

$$\frac{x}{6} + \frac{y}{4} = 1$$

$$\Rightarrow 4x + 6y = 24$$

$$\Rightarrow 2x + 3y = 12$$

- **Normal form of the equation of a line**

The equation of the line at normal distance p from the origin and angle ω , which the normal makes with the positive direction of the x -axis is given by $x \cos \omega + y \sin \omega = p$

Example: Reduce the equation $x - \sqrt{3}y - 6 = 0$ to normal form and hence find the length of perpendicular to the line from the origin. Also find angle between the normal and positive direction of the x -axis.

Solution: The given equation is $x - \sqrt{3}y - 6 = 0$.

$$\Rightarrow x - \sqrt{3}y = 6 \quad \dots(1)$$

On dividing (1) by $\sqrt{(\sqrt{1})^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$, we obtain

$$\frac{1}{2}x - \frac{\sqrt{3}}{2}y = 3$$



$$\Rightarrow x \cos 300^\circ + y \sin 300^\circ = 3 \quad \dots (2)$$

On comparing equation (2) with $x \cos \omega + y \sin \omega = p$, we obtain $\omega = 300^\circ$ and $p = 3$

Therefore, the length of perpendicular to the line from the origin is 3 units and the angle between the normal and the positive x -axis is 300° .

Parametric form

The equation of a line in parametric form is $x - x_1 \cos \theta = y - y_1 \sin \theta = r$.

Also, from the above equation, we get:

$$x - x_1 = r \cos \theta \text{ and } y - y_1 = r \sin \theta \therefore x = x_1 + r \cos \theta \text{ and } y = y_1 + r \sin \theta$$

Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $x_1 + r \cos \theta$, $y_1 + r \sin \theta$.

- **Angle between two lines:** An acute angle, θ , between line l_1 and l_2 with slopes m_1 and m_2 respectively is given by

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|, \quad 1 + m_1 m_2 \neq 0$$

Example 1: Two lines AB and CB, intersect at point B. The coordinates of end points are A(-4, -3), B(0, 5), and C(10, 5). Find the measures of angles between AB and CB.

Solution: Let the angle between the lines AB and BC be θ .

$$\text{Slope of line AB} = \frac{5 - (-3)}{0 - (-4)} = \frac{8}{4} = 2$$

$$\text{Slope of line BC} = \frac{5 - 5}{10 - 0} = 0$$

We know that the angle between two lines with slopes m_1 and m_2 is given by $\tan \theta = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|$.

$$\text{Therefore, } \tan \theta = \left| \frac{2 - 0}{1 + 2 \times 0} \right| = 2$$

$$\Rightarrow \theta = \tan^{-1}(2).$$

Point of Intersection of Two Intersecting Lines

- If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ are two intersecting lines, then the coordinates of their point of intersection are

$$\frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}, \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}, \text{ where } a_1 b_2 - a_2 b_1 \neq 0.$$

- Three or more distinct lines are said to be concurrent if they pass through a common point, and the point of intersection is called the point of concurrence.



- If the three lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ are concurrent, then $a_1b_2c_3-b_1c_2c_3-a_3c_2+c_1a_2b_3-a_3b_2=0$.

• Distance of a Point From a Line

The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Example: Find the distance of point $(1, -2)$ from the line $8x - 6y - 12 = 0$.

Solution: On comparing the equation of the given line i.e., $8x - 6y - 12 = 0$ with $Ax + By + C = 0$, we obtain $A = 8$, $B = -6$, $C = -12$

The distance (d) of point $(1, -2)$ from line $8x - 6y - 12 = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|8 \times 1 + (-6)(-2) + (-12)|}{\sqrt{8^2 + 6^2}} = \frac{|8 + 12 - 12|}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$

• Distance between parallel lines

The distance (d) between two parallel lines i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by,

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

Example: Find the distance between the lines $4x + 3y = 11$ and $4x + 3y = 8$.

Solution: The given lines are $4x + 3y - 11 = 0$ and $4x + 3y - 8 = 0$

Slope of the line $4x + 3y - 11 = 0$ is $-\frac{4}{3}$.

Slope of the line $4x + 3y - 8 = 0$ is $-\frac{4}{3}$.

Since the slopes of the given lines are equal, the lines are parallel.

Here, $A = 4$, $B = 3$, $C_1 = -11$ and $C_2 = -8$

$$\text{Distance between the lines} = \left| \frac{-11 - (-8)}{\sqrt{4^2 + 3^2}} \right| = \left| \frac{-11 + 8}{\sqrt{16 + 9}} \right| = \left| \frac{-3}{\sqrt{25}} \right| = \frac{3}{5}$$

Family of Lines

If $u \equiv a_1x + b_1y + c_1 = 0$ and $v \equiv a_2x + b_2y + c_2 = 0$ are two intersecting lines, then the equation $u + \lambda v = 0$ represents a family of lines, each passing through the point of intersection of the lines $u = 0$ and $v = 0$, where λ is a parameter.

